Subject Name: **Theory of machines & mechanics**

Subject Code: **ME-3003**

Semester: **3rd**

Unit-V

Gyroscope: Gyroscopic Action in Machines, Angular Velocity and Acceleration, Gyroscopic torque/ couple, the Gyroscopic effect on Naval Ships, Stability of Two and Four Wheel Vehicles, Rigid disc at an angle fixed to a rotating shaft.

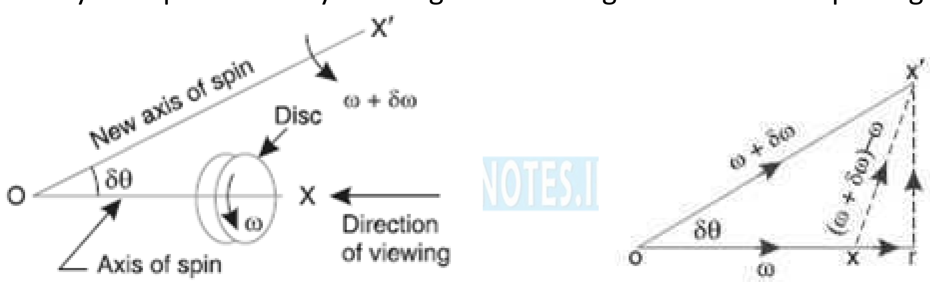
Introduction

When a body is moving in a curved path with a uniform linear velocity, a force in the direction of centripetal acceleration (known as centripetal force) has to be applied externally over the body, so that it moves along the required curved path. This external force applied is known as active force.

When a body is moving with uniform linear velocity along a circular path by itself, it is subjected to the centrifugal force radially outwards. This centrifugal force is called reactive force. The action of the reactive or centrifugal force is to tilt or move the body along the radially outward direction.

Processional Angular Motion

We know that the rate of change of angular velocity with respect to time is known as angular acceleration. It is a vector quantity and may be represented by drawing a vector diagram with the help of right-hand screw rule.



(a) (b)

Fig. 5.1 Processional angular motion

Consider a disc, as shown in Fig. 5.1. (a), revolving or spinning about the axis O X (known as the **axis of spin**) in anticlockwise when seen from the front, with an angular velocity ω in a plane at right angles to the paper.

After a short interval of time δ t, let the disc be spinning about the new axis of spin OX ′ (at an angle δθ) with an angular velocity (ω + δω). Using the right-hand screw rule, the initial angular velocity of the disc (ω) is represented by vector ox; and the final angular velocity of the disc (ω + δω) is represented by vector ox’ as shown in Fig. 5.1 (b). The vector x x’ represents the change of angular velocity in time δ t i.e. the angular acceleration of the disc. This may be resolved into two components, one parallel to the ox and the other perpendicular to the ox.

Component of angular acceleration in the direction of ox,

$\alpha_t = xr/\delta t = (or - ox)/ \delta t = (ox' \cos \delta\theta - ox) / \delta t$

$\alpha_t = \{(\omega + \delta\omega) \cos \delta\theta - \omega \}/\delta t = (\omega \cos \delta\theta + \delta\omega \cos \delta\theta - \omega) / \delta t$

Since δθ is very small, therefore substituting cos δθ = 1, we have

$\alpha_t = (\omega\ \delta\theta + \delta\omega.\ \delta\theta - \omega) / \delta t = \delta\omega / \delta t$

In the limit, when $\delta t \rightarrow 0$

$\alpha_t = Lt_{\delta t \rightarrow 0} (\delta\omega / \delta t) = d\omega / dt$

Component of angular acceleration in the direction perpendicular to ox,

$\alpha_e = rx'/\delta t = \{ox' \sin \delta\theta\}/\delta t =\{ (\omega + \delta\omega) \sin \delta\theta\}/ \delta t = (\omega \sin \delta\theta + \delta\omega \sin \delta\theta) / \delta t$

Since $\delta\theta$ in very small, therefore substituting $\sin \delta\theta = \delta\theta$, we have
$\alpha_e = (\omega.\ \delta\theta + \delta\omega.\ \delta\theta) / \delta t = \omega.\ \delta\theta / \delta t$
In the limit when $\delta t \rightarrow 0$,
$\alpha_e = Lt_{\delta t \rightarrow 0} (\omega.\ \delta\theta / \delta t) = \omega.\ \delta\theta / dt = \omega.\ \omega_P$
∴ Total angular acceleration of the disc
= vector xx′ = vector sum of α_t and α_c
$= d\omega / dt + \omega.\ \delta\theta / dt = d\omega / dt + \omega.\ \omega_P$
where $d\theta/dt$ = the angular velocity of the axis of spin about a certain axis, which is perpendicular to the plane in which the axis of spin is going to rotate. This angular velocity of the axis of spin (i.e. $d\theta/dt$) is known as **angular velocity of precession** and is denoted by ω_P. The axis, about which the axis of spin is to turn, is known as the **axis of precession.** The angular motion of the axis of spin about the axis of precession is known as **precessional angular motion.**

Gyroscopic Couple

Consider a disc spinning with an angular velocity ω rad/s about the axis of spin OX, in an anticlockwise direction when seen from the front, as shown in Fig. 5.2 (a). Since the plane in which the disc is rotating is parallel to the plane YOZ, therefore it is called **plane of spinning.** The axis of spin rotates in a plane parallel to the horizontal plane about an axis OY and the horizontal plane is the plane XOZ. In other words, the axis of spin is said to be rotating or processing about an axis OY. In other words, we can say that the axis of spin is said to be rotating or processing about an axis OY (which is perpendicular to both the axes OX and OZ) at an angular velocity ωP rap/s. This horizontal plane XOZ is called **plane of precession** and OY is the **axis of precession.**
Let I is the Mass moment of inertia of the disc about OX, and
ω = Angular velocity of the disc.
∴ Angular momentum of the disc
= I.ω
Since the angular momentum is a vector quantity, therefore it may be represented by the vector →ox shown in Fig. 5.2 (b). When seen from the top about the axis OY the axis of spin is also rotating in an anticlockwise direction. Let the axis OX is turned in the plane XOZ through a small angle δθ radians to the position OX′, in time δt seconds. Let us assume that the angular velocity ω will be constant, the angular momentum will now be represented by vector ox′.

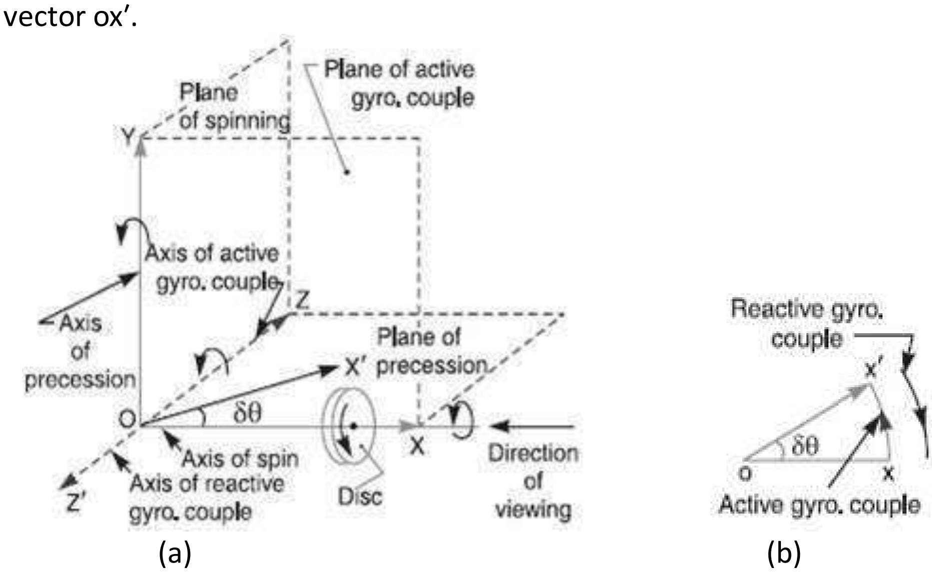


Fig. 5.2 Gyroscopic couple

∴ Change in angular momentum
= I. ω.δθ
and rate of change of angular momentum
= I. ω.δθ/dt

AS we know that the application of a couple of the disc results in the rate of change of angular momentum, therefore the couple applied to the disc causing precession,

$C = Lt_{\delta t \to 0}\ I.\ \omega.\delta\theta/dt = I.\ \omega \times \delta\theta/dt = I.\ \omega.\ \omega_P$

where ωP = Angular velocity of the precession of the axis of spin or the speed of rotation of the axis of spin about the axis of precession OY. The S.I. unit of C is N-m when I is in kg-m^2.

It may be noted that

1. The couple $I.\omega.\omega_p$, in the direction of the vector xx′ (representing the change in angular momentum), is the **active gyroscopic couple,** which has to be applied over the disc when the axis of spin is made to rotate with angular velocity ω_P about the axis of precession. The vector xx′ lies in the plane XOZ or the horizontal plane. In case of a very small displacement δθ, the vector xx′ will be perpendicular to the vertical plane XOY. Thus couple will lie in the plane XOY which causing this change in the angular momentum. The vector xx′, as shown in Fig. 5.2 (b), represents an anticlockwise couple in the plane XOY. Therefore, the plane XOY is called the **plane of the active gyroscopic couple** and the axis OZ perpendicular to the plane XOY, about which the couple acts, is called the axis of the active gyroscopic couple.

2. When the axis of spin itself moves with angular velocity ω_P, the disc is subjected to a **reactive couple** whose magnitude is same (i.e. $I.\ \omega.\omega_P$) but opposite in direction to that of the active couple. This reactive couple to which the disc is subjected when the axis of spin rotates about the axis of precession is known as a **reactive gyroscopic couple.** The axis of the reactive gyroscopic couple is represented by OZ′ in Fig. 5.2 (a).

3. The gyroscopic couple is usually applied through the bearings which support the shaft. The bearings will resist equal and opposite couple.

4. The gyroscopic principle is used in an instrument or toy known as a **gyroscope.** To minimize the rolling and pitching effects of waves the gyroscopes are installed in ships. They are also used in aeroplanes, monorail cars, gyrocompasses etc.

Effect of the Gyroscopic Couple on an Aeroplane

The top and front view of an aeroplane is shown in Fig. 5.3 (a). Let us assume that the engine or propeller rotates in the clockwise direction when seen from the rear or tail end and the aeroplane takes a turn to the left.

Let ω is equal to the Angular velocity of the engine in rad/s,

m is equal to the Mass of the engine and the propeller in kg,

k is the radius of gyration in metres of the engine,

I is the Mass moment of inertia of the engine and the propeller in kg-$m^2 = m.k^2$,

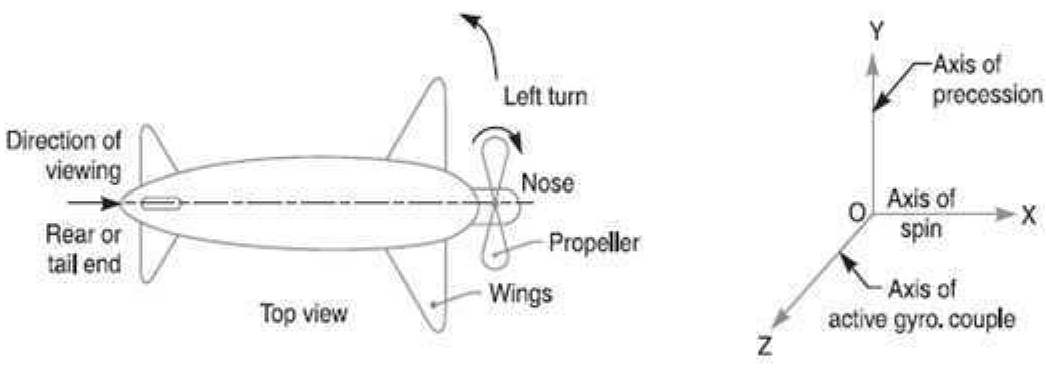
v is the Linear velocity of the aeroplane in m/s,

R is the Radius of curvature in metres, and

ω_P = Angular velocity of precession = v/R rad/s

∴ Gyroscopic couple acting on the aeroplane,

$C = I.\omega.\omega_P$



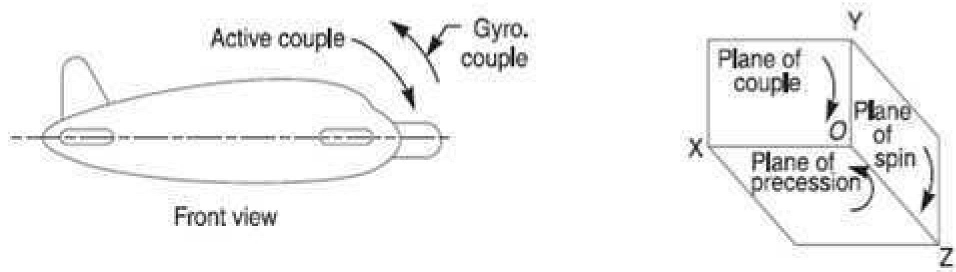
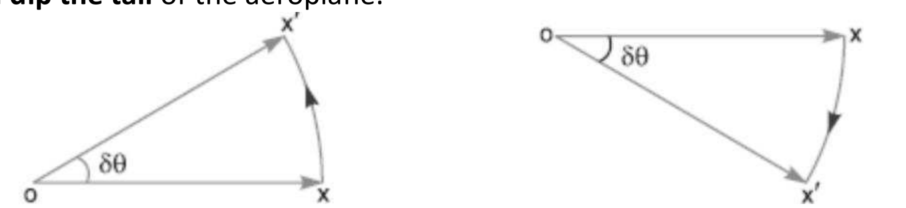


Fig. 5.3 Aeroplane taking a left turn

Before taking the left turn, the angular momentum vector is represented by ox. When an aeroplane takes a left turn, the direction of the angular momentum vector will change from ox to ox′ due to the active gyroscopic couple as shown in Fig. 5.4 (a). The vector xx′, in the limit, represents the change of angular momentum or the active gyroscopic couple and is perpendicular to the ox. Therefore the plane of an active gyroscopic couple (XOY) will be perpendicular to xx′, i.e. vertical in this case, as shown in Fig. 5.4 (b). By applying right-hand screw rule to vector xx′, we find that the direction of the active gyroscopic couple is clockwise as shown in the front view of Fig. 5.4 (a). In other words, the active gyroscopic couple on the aeroplane in the axis OZ will be clockwise for left-hand turning as shown in Fig. 5.4 (b).The reactive gyroscopic couple (equal in magnitude of the active gyroscopic couple) will act in the anticlockwise direction and the effect of this couple is, therefore, to **raise the nose** and **dip the tail** of the aeroplane.



(a) Aeroplane taking left turn (b) Aeroplane taking a right turn.

Fig. 5.4 Effect of the gyroscopic couple on an aeroplane

Terms Used in a Naval Ship

The top and front views of a naval ship are shown in Fig. 5.5. The fore end of the ship is called **bow** and the rear end is known as **stern** or **aft**. The left and right-hand side of the ship is called **port** and **starboard** respectively when viewed from the stern. We shall now discuss the effect of the gyroscopic couple on the naval ship in the following three cases:

1. Steering, **2.** Pitching, and **3.** Rolling.

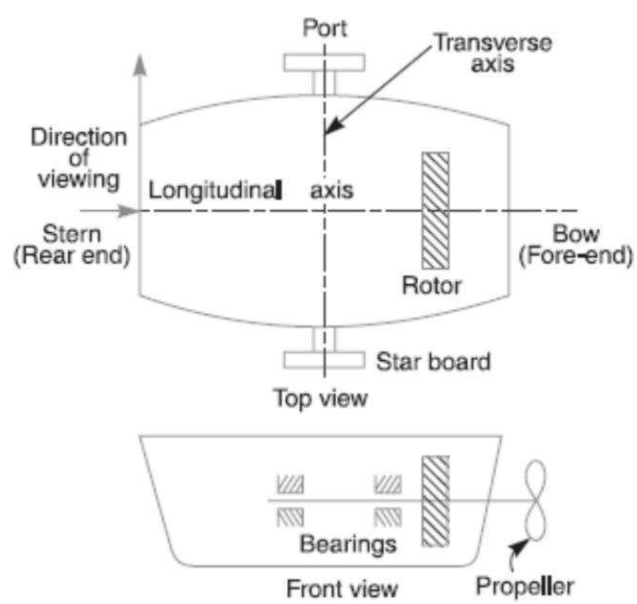


Fig. 5.5 Terms used in a naval ship

1. Effect of Gyroscopic Couple on a Naval Ship during Steering

Steering is the turning of a complete ship in a curve towards left or right, while it moves forward. Consider the ship taking a left turn, and the rotor rotates in the clockwise direction when viewed from the stern, as shown in Fig. 5.6. The effect of the gyroscopic couple on a naval ship during steering taking left or right turn may be obtained in a similar way as for an aeroplane.

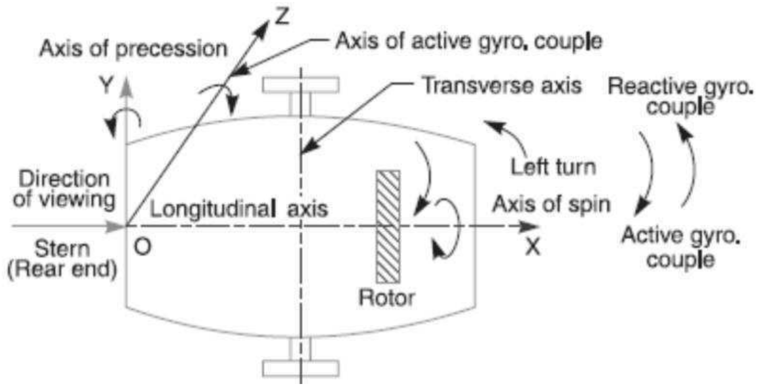


Fig. 5.6 The Naval ship taking a left turn.

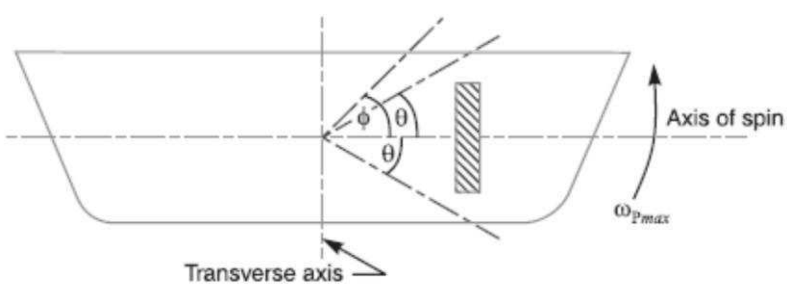
When the rotor of the ship rotates in the clockwise direction when viewed from the stern, it will have its angular momentum vector in the direction ox as shown in Fig. 5.7 (a). As the ship steers to the left, the angular momentum vector will change from ox to ox’ due to the active gyroscopic couple, the active gyroscopic couple will change the angular momentum vector from ox to ox’. The vector xx’ now represents the active gyroscopic couple and is perpendicular to the ox. Thus the plane of the active gyroscopic couple is perpendicular to xx’ and its direction in the axis OZ for a left-hand turn is clockwise as shown in Fig. 5.7. The reactive gyroscopic couple of the same magnitude will act in the opposite direction (i.e. in the anticlockwise direction). The **effect of this reactive gyroscopic couple is to raise the bow and lower the stern.**

(a) Steering to the Left (b) Steering to the Right

Fig. 5.7 Effect of gyroscopic couple on a naval ship during steering

2. Effect of Gyroscopic Couple on a Naval Ship during Pitching

Pitching is the movement of a complete ship up and down in a vertical plane about a transverse axis, as shown in Fig. 5.8 (a). In this case, the transverse axis becomes the axis of precession and the pitching of the ship is assumed to take place with simple harmonic motion i.e. the motion of the axis of spin about the transverse axis is simple harmonic.



(a) Pitching of a Naval Ship

(b) Pitching Upward (c) Pitching Downward

Fig. 5.8 Effect of gyroscopic couple on a naval ship during pitching

∴ Angular displacement of the axis of spin from mean position after time t seconds,
$\theta = \phi \sin \omega_1 . t$
where ϕ = Amplitude of swing i.e. maximum angle turned from the mean position in radians, and
ω_1 = Angular velocity of S.H.M.
$\omega_1 = 2\pi$ / Time Period of S. H. M. in seconds $= 2\pi/t_P$ rad/s
Angular velocity of precession,
$\omega_P = d\theta/dt = d/dt\,(\Phi \sin \omega_1 . t) = \Phi . \omega_1 . \cos \omega_1 t$
The angular velocity of precession will be maximum if $\cos \omega_1 . t = 1$.
∴ Maximum angular velocity of precession,
$\omega_{Pmax} = \phi . \omega_1 = \phi \times 2\pi / t_p$
Let I = Moment of inertia of the rotor in kg-m^2, and
ω = Angular velocity of the rotor in rad/s.
∴ Maximum gyroscopic couple,
$C_{max} = I . \omega . \omega_{Pmax}$
When the pitching is upward, the effect of the reactive gyroscopic couple, as shown in Fig. 5.8 (b), will try to move the ship toward starboard. On the other hand, if the pitching is downward, the effect of the reactive gyroscopic couple, as shown in Fig. 5.8 (c), is to turn the ship towards port side.

3. Effect of Gyroscopic Couple on a Naval Ship during Rolling

We know that, for the effect of the gyroscopic couple to occur, the axis of precession should always be perpendicular to the axis of spin. There will be no effect of the gyroscopic couple acting on the body of the ship when the axis of precession becomes parallel to the axis of spin. In case of rolling of a ship, the axis of precession (i.e. longitudinal axis) is always parallel to the axis of spin for all positions. Hence, there will be no effect of the gyroscopic couple acting on the body of a ship.

Stability of a Four Wheel Drive Moving in a Curved Path

Consider the four wheels A, B, C and D of an automobile locomotive taking a turn towards left as shown in Fig. 5.9. The inner wheels are A and C, whereas the outer wheels are B and D. The centre of gravity (C.G.) of the vehicle lies vertically above the road surface.
Let m = Mass of the vehicle in kg,
W = Weight of the vehicle in newtons = m.g,
r_W = Radius of the wheels in metres,
R = Radius of curvature in metres ($R > r_W$),
h = Distance of centre of gravity, vertically above the road surface in metres,
x = Width of track in metres,
I_W = Mass moment of inertia of one of the wheels in kg-m^2,
ω_W = Angular velocity or velocity of spin of the wheels in rad/s,
I_E = Mass moment of inertia of the rotating parts of the engine in kg-m^2,
ω_E = Angular velocity of the rotating parts of the engine in rad/s,
G = Gear ratio = ω_E / ω_W,
v = Linear velocity of the vehicle in m/s = $\omega_W . r_W$

Here let us consider that the weight of the vehicle (W) will be equally distributed over the four wheels which will act downwards. The reaction between each wheel and the road surface of the same magnitude will act upwards.
Therefore
Road reaction over each wheel = $W/4 = m.g/4$ newtons
Now let us consider the effect of the gyroscopic couple and centrifugal couple on the vehicle.

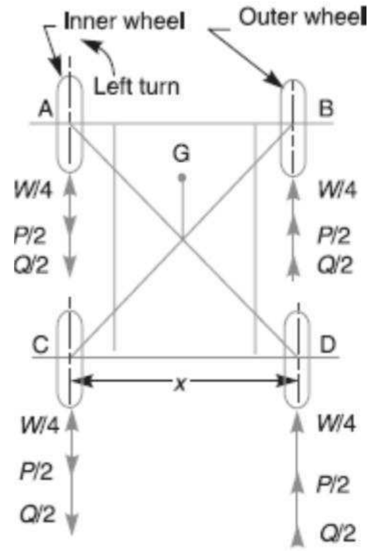


Fig. 5.9 Four-wheel drive moving in a curved path

1. Effect of the gyroscopic couple
Since the vehicle is taking a turn towards the left due to the precession and other rotating parts, therefore a gyroscopic couple will act.
We know that velocity of precession,
$\omega_P = v/R$
∴ Gyroscopic couple due to 4 wheels,
$C_W = 4\, I_W.\omega_W.\omega_P$
and a gyroscopic couple due to the rotating parts of the engine,
$C_E = I_E.\omega_E.\omega_P = I_E.G.\omega_W.\omega_P$ ... $(G = \omega_E/\omega_W)$
∴ Net gyroscopic couple,
$C = C_W \pm C_E = 4\, I_W.\omega_W.\omega_P \pm I_E.G.\omega_W.\omega_P = \omega_W.\omega_P\,(4\, I_W \pm G.I_E)$
The **positive** sign is used when the wheels and rotating parts of the engine rotate in the same direction. If the rotating parts of the engine revolve in opposite direction, then the **negative** sign is used.
Due to the gyroscopic couple, the vertical reaction on the road surface will be produced. The reaction on the outer wheels will be vertically upwards and vertically downwards on the inner wheels.
Let the magnitude of this reaction at the two outer or inner wheels be P newtons. Then
$P \times x = C$ or $P = C/x$
∴ Vertical reaction at the outer or inner wheels,
$P/2 = C/2x$

Effect of the centrifugal couple
Since the vehicle moves along a curved path, therefore centrifugal force will act outwardly at the centre of gravity of the vehicle. The effect of this centrifugal force is also to overturn the vehicle.
We know that centrifugal force,
$F_C = m \times v^2 / R$
∴ The couple tending to overturn the vehicle or overturning couple,
$C_O = F_C \times h = m \times v^2 \times h / R$

This overturning couple which is vertically upwards on the outer wheels and vertically downwards on the inner wheels is balanced by vertical reactions, Let Q be the magnitude of this reaction at the two outer or inner wheels. Then

$Q \times x = C_O$ or $Q = C_O / x = m \times v^2 \times h / R.x$

∴ Vertical reaction at each of the outer or inner wheels,

$Q/2 = m \times v^2 \times h / 2R.x$

∴ Total vertical reaction at each of the outer wheel,

$P_O = W/4 + P/2 + Q/2$

And total vertical reaction at each of the inner wheel,

$P_1 = W/4 - P/2 - Q/2$

A little consideration will show that when the vehicle is running at high speeds, PI may be zero or even negative. It will cause the inner wheels to leave the ground thus tending to overturn the automobile the sum of P/2 and Q/2 must be less than W/4, in order to maintain the contact between the inner wheels and the ground, the sum of P/2 and Q/2 must be less than W/4.

Stability of a Two Wheel Vehicle Taking a Turn

Consider a two-wheel vehicle (say a scooter or motorcycle) taking a right turn as shown in Fig. 5.10 (a)

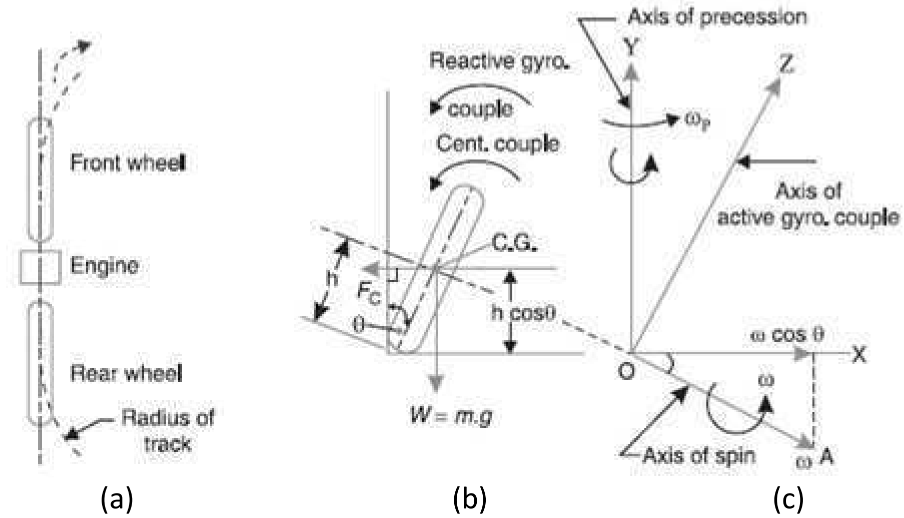


Fig. 5.10 Stability of a two-wheel vehicle taking a turn

Let m will be the Mass of the vehicle and its rider in kg,

W = Weight of the vehicle and its rider in newtons = m.g,

h = Height of the centre of gravity of the vehicle and rider,

r_W = Radius of the wheels,

R = Radius of track or curvature,

I_W = Mass moment of inertia of each wheel,

I_E = Mass moment of inertia of the rotating parts of the engine,

ω_W = Angular velocity of the wheels,

ω_E = Angular velocity of the engine,

G = Gear ratio = ω_E / ω_W,

v = Linear velocity of the vehicle = $\omega W \times rW$,

θ = Angle of the heel. It is the inclination of the vehicle to the vertical for equilibrium.

Let us now consider the effect of the gyroscopic couple and centrifugal couple on the vehicle, as discussed below.

1. Effect of gyroscopic couple

We know that

$v = \omega_W \times r_W$ or $\omega_W = v / r_W$

$\omega_E = G\ \omega_E = G \times v / r_W$

$\therefore$ Total $(I \times \omega) = 2\ I_W \times \omega_W \pm I_E \times \omega_E$

$= 2\ I_W \times v/r_W \pm I_E \times G \times v/r_W = v/r_W\ (2\ I_W \pm G.\ I_E)$

And velocity of precession, $\omega P = v / R$

Let us consider that when the wheels move over the curved path, the vehicle is always inclined at an angle θ with the vertical plane as shown in Fig. 5.10 (b). This angle is known as the **angle of heel.** In other words, the axis of spin is inclined to the horizontal at an angle θ, as shown in Fig. 5.10 (c). Thus the angular momentum vector Iω due to spin is represented by OA inclined to OX at an angle θ. But the precession axis is vertical. Therefore the spin vector is resolved along OX.

$\therefore$ Gyroscopic couple,

$C_1 = I.\ \omega \cos\theta \times \omega P = v/r_W\ (2\ I_W \pm G.\ I_E) \cos\theta \times v/R$

$= v^2/R.\ r_W\ (2\ I_W \pm G.\ I_E) \cos\theta$

2. Effect of centrifugal couple

We know that centrifugal force,

$F_C = m \times v^2 / R$

This force acts horizontally through the centre of gravity (C.G.) along the outward direction.

$\therefore$ Centrifugal couple,

$C_2 = F_C \times h \cos\theta = (m \times v^2/R)\ h \cos\theta$

Since the centrifugal couple has a tendency to overturn the vehicle, therefore

Total overturning couple,

C_O = Gyroscopic couple + Centrifugal couple

$C_O = v^2/R.\ r_W\ (2\ I_W + G.\ I_E) \cos\theta + (m \times v^2/R)\ h \cos\theta$

$C_O = v^2/R\ [(2\ I_W + G.\ I_E)/\ r_W + m.\ h] \cos\theta$

We know that balancing couple = m.g.h sin θ

The balancing couple acts in a clockwise direction when seen from the front of the vehicle.

Therefore for stability, the overturning couple must be equal to the balancing couple, i.e.

$v^2/R\ [(2\ I_W + G.\ I_E)/\ r_W + m.\ h] \cos\theta = m.g.h \sin\theta$

From the above expression, the value of the angle of heel (θ) may be determined, so that the vehicle does not skid.

Effect of Gyroscopic Couple on a Disc Fixed Rigidly at a Certain Angle to a Rotating Shaft

Consider a disc fixed rigidly to a rotating shaft such that the polar axis of the disc makes an angle θ with the shaft axis, as shown in Fig. 5.11. Let the shaft rotates with an angular velocity ω rad/s in the clockwise direction when viewed from the front. Let us consider that the disc will also rotate about OX with the same angular velocity ω rad/s. Let OP be the polar axis and OD the diametral axis of the disc.

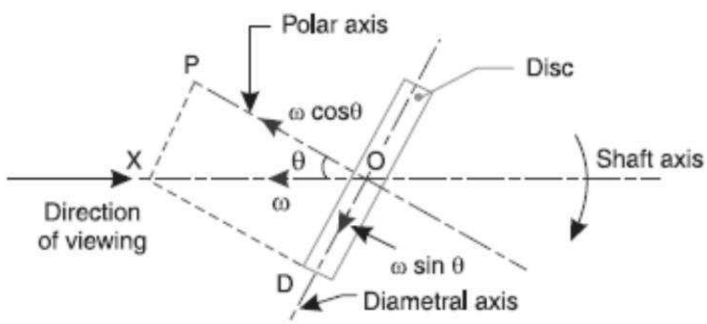


Fig. 5.11 Effect of gyroscopic couple on a disc fixed rigidly at a certain angle to a rotating shaft

$\therefore$ The angular velocity or the angular velocity of spin of the disc about the polar axis OP

= ω cos θ ... (Component of ω in the direction of OP)

Since the shaft rotates, therefore the point P will move in a plane perpendicular to the plane of the paper. In other words, precession is produced about OD.

∴ The angular velocity of precession or the angular velocity of the disc about the diametral axis OD = $\omega \sin\theta$

If IP is the mass moment of inertia of the disc about the polar axis OP, then gyroscopic couple acting on the disc,

$C_P = I_P . \omega \cos\theta . \omega \sin\theta = ½ \times I_P . \omega^2 \sin 2\theta$ ... (∵ $2 \sin\theta \cos\theta = \sin 2\theta$)

The effect of this gyroscopic couple is to turn the disc in the anticlockwise when viewed from the top, about an axis through O in the plane of the paper.

Now consider the movement of point D about the polar axis OP. In this case, OD is the axis of spin and OP is the axis of precession.

∴ Angular velocity of disc about OD or angular velocity of spin = $\omega \sin\theta$

And angular velocity of precession or the angular velocity of D about OP = $\omega \cos\theta$

If ID is the mass moment of inertia of the disc about the diametral axis OD, then gyroscopic couple acting on the disc,

$C_D = I_D . \omega \sin\theta . \omega \cos\theta = ½ \times I_D . \omega^2 \sin 2\theta$

The effect of the couple in the above expression will be opposite to that of CP.

∴ Resultant gyroscopic couple acting on the disc,

$C = C_P – C_D = ½ \times \omega^2 \sin 2\theta (I_P - I_D)$

This resultant gyroscopic couple will act in the anticlockwise direction as seen from the top. In other words, the shaft tends to turn in the plane of the paper in an anticlockwise direction as seen from the top, as a result the horizontal force is exerted on the shaft bearings.

